Math Runes

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Abstract

We examine a scheme to create simple patterns of lines or arcs within circles which represent mathematic functions. These delightful patterns contain splendid symmetries and relationships that invite mathematical exploration. Math runes can be used in the classroom with pupils of all ages to learn about functions and representations and to inspire artistic and mathematical wonder.

Introduction

When we create artwork, we can learn a lot about mathematics. We can also use mathematics to make us better artists. These two opposite and equal ideas are the foundation for mathematical art classes that I have taught to elementary and middle school pupils and teachers for several years. Activities include paper folding and cutting, building sculptures and drawing and coloring designs that are based on mathematical patterns and principles.

Creating "Math Runes" (as I call them) is one such activity. These runes are beautiful and interesting patterns that offer rich hunting grounds for mathematical patterns and ideas. Furthermore, making the patterns is fun, rewarding and inspiring.

Runes are letters in a set of alphabets that were used in Europe from about 150 AD to 1100 AD. Several runes from the Scandanavian Elder Futhark alphabet (also called Viking runes) are shown in Figure 1.

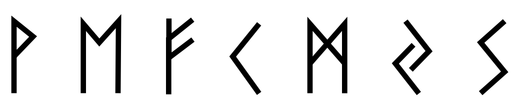


Figure 1: Viking runes

The shapes are interesting and could themselves be used for geometric exploration and artwork, but we shall make our own set of runes with mathematical structure of our own choosing.

Let us begin our journey with the Elder Futhark rune *laguz*, which symbolizes the beginning of a journey.

Math Runes

These splendid forms contain delightful patterns and surprises. Because they are based on mathematical functions and relationships, they are connected to a host of fascinating number properties. Math runes are drawn by connecting pairs of points on a circle. The circle has 10 equally spaced points numbered 0-9, and the pairs are determined by applying a function to each number from 0-9 and connecting that point to the point with the number which matches the result. Results greater than 9 “wrap around” the circle (modulo 10), so a result of 35 for example indicates point 5. This is also the last digit of 35, so it is very easy to find the correct point for any result in modulo 10.

Any function that produces integer results from the inputs 0-9 works. We’ll begin with a group of simple multiplication runes, share some of the mathematical ideas and examples of pupils’ work, and then provide both simpler and more complex versions of math runes so that the activity can be used successfully with pupils at all levels from age six to adult.

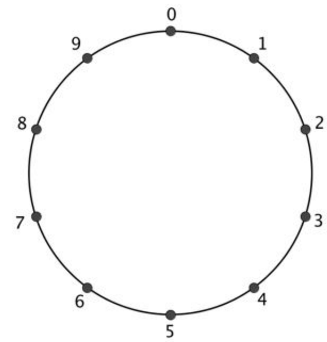
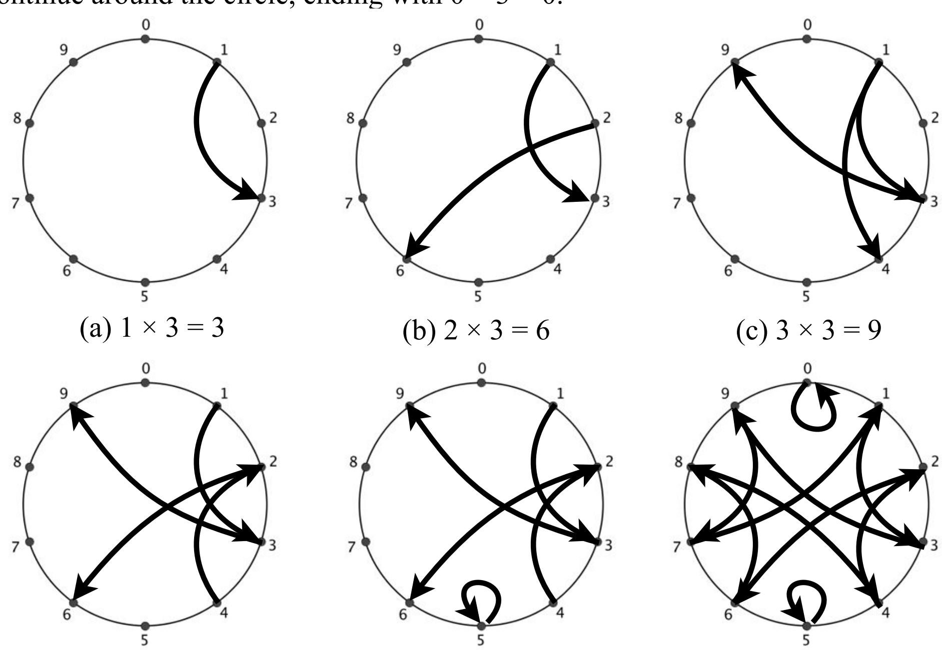


Figure 2: Math rune template

Multiplication Runes

Here’s how to construct the rune for “multiply by 3.”

(a) Start with 1 as the input and apply the rule. 1 × 3 = 3. Connect 1 (the input) with 3 (the output). Arcs look nice, but straight line segments work well also. Arrows may be used to indicate direction. The style is up to the artist!
(b) The next point, 2, connects to point 6, because 2 × 3 = 6.
(c) Likewise 3 connects to 9.
(d) 4 × 3 = 12, so we travel all the way around the circle and end up at point 2. This is 12 mod 10, or simply the last digit of the output.
(e) 5 × 3 = 15, so 5 connects with itself. This can be indicated by a loop (or some other manner).
(f) Continue around the circle, ending with 0 × 3 = 0.



(a) 1 × 3 = 3
(b) 2 × 3 = 6
(c) 3 × 3 = 9
(d) 4 × 3 = 12 . . . 12 mod 10 = 2
(e) 5 × 3 = 15 . . . 15 mod 10 = 5
(f) completed rune

Figure 3: Constructing the “times 3” rune

The “times 3” rune has reflective symmetry. Will other multiplication runes have symmetry as well? What kinds of similarities and differences are there between the runes? The pupils in my class made interesting discoveries, including: “times 3” and “times 7” look the same; zero always connects to zero; five connects to zero if it is an “even rune”, to five if it is an “odd rune”; all figures have a vertical line of symmetry and odd runes have a horizontal line of symmetry as well.

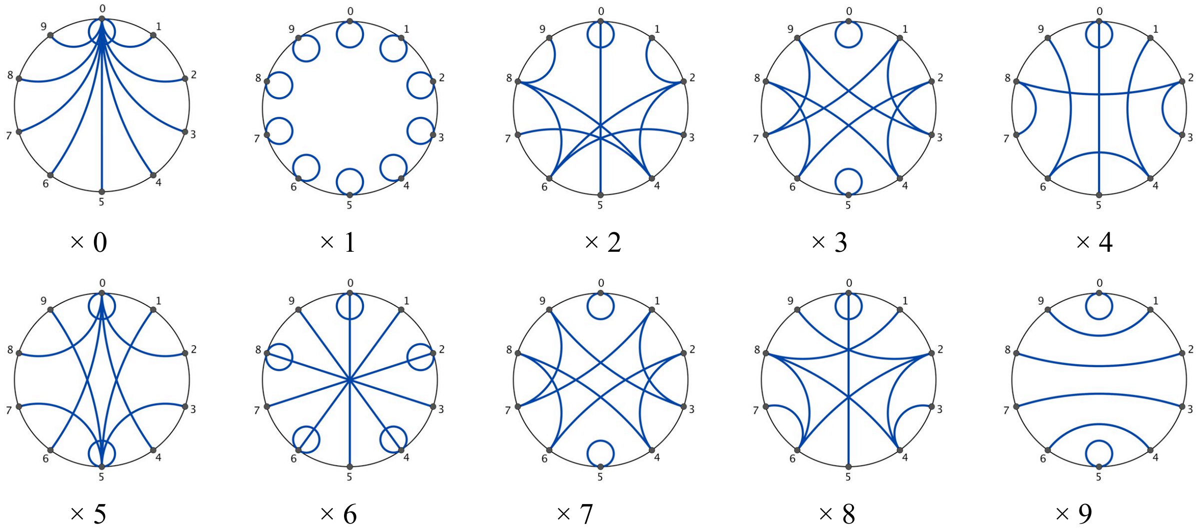


Figure 4: Multiplication Runes from “× 0” to “× 9”

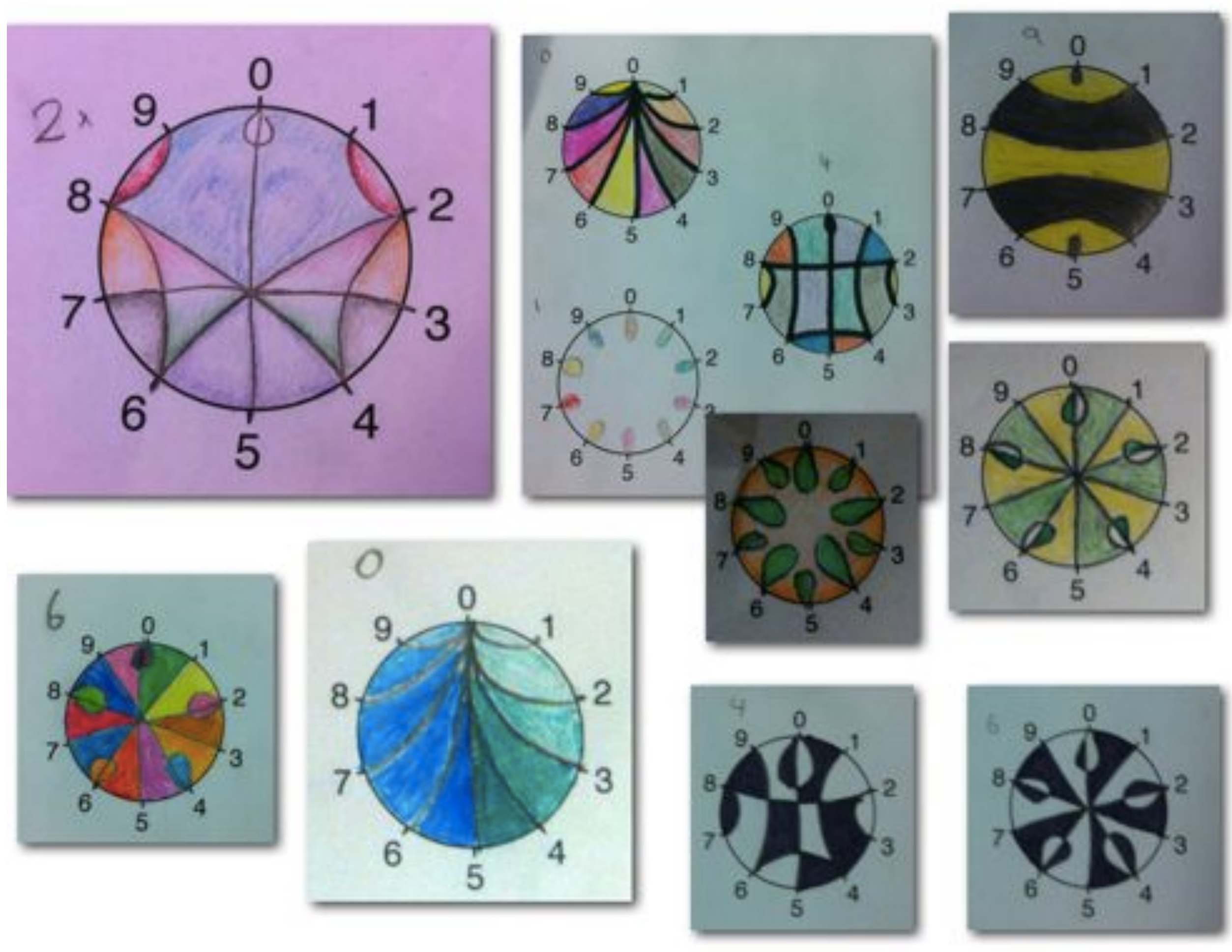


Figure 5: 6th grade pupil-created runes

After discussing our initial discoveries, we allowed ample time to color the runes and explore further. Some pupils' artwork is shown in Figure 5. Some forms were colored according to ideas suggested by the shapes: one girl thought that the "times 9" rune resembled a bumblebee, and so colored her design black and yellow. Plants, flowing water, and the earth were also recurring themes. Coloring the runes also gave children time to think about the shapes and in some cases deepen their understanding of the number patterns. Taking time to play with ideas is an important part of learning that is too often missing from mathematics classroom; creating art can provide this opportunity and its value should not be underestimated!

The identical-looking runes for "times 3" and "times 7" were particularly tantalizing for several pupils, and the fact that 3+7=10 caused them to compare other pairs of runes to see if there were some sort of way to naturally pair the runes. This led to close scrutiny of attributes of the figures and some exciting observations. Runes 0 and 5 seem to pair naturally together, as do 1 and 9. Runes 2 and 8 have some tantalizing similarities, but why then do the runes for 4 and 6 seem worlds apart? Is there a common way all of these pairs are related? In fact there is: take each pair of arcs in a run that are relections of each other along the vertical axis and swap one pair of endpoints. The rune transforms into its partner! It is interesting to examine the runes with this in mind, think about what is happening numerically, and also about what makes 3 and 7 so special. There are fantastic things happening even in something as simple as muliplying one-digit numbers!

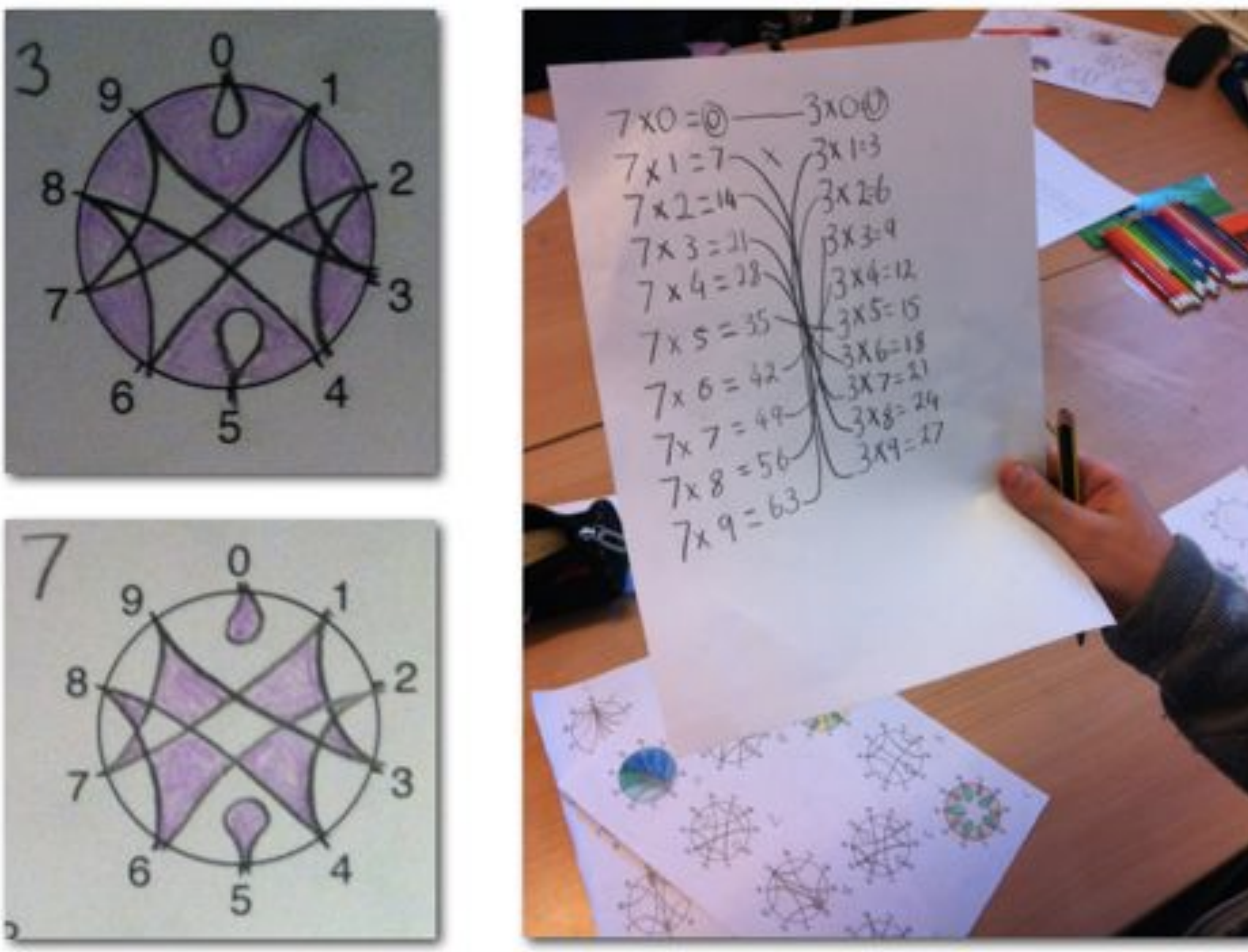


Figure 6: An 11-year old's explanation of why the "times 3" and "times 7" runes are identical

My pupils were eager to continue and asked if I thought it would work with addition or subtraction as well. (My answer to these questions is almost always "I don't know! Should we try?") We will now examine some of these other families of runes with both simpler and more complicated functions.

Addition Runes

An easier set of runes can be made using addition instead of multiplication. The “plus 2” rune, for example, is made by connecting each point with the point whose value is 2 greater (“wrapping around” from 9 to 0, of course). Each of these runes has a star-like pattern. Pupils are generally surprised as they discover their first star, and many quickly predict that the other runes in the group will also be star-like. Some pupils are able to visualize the other runes before they draw them and predict which runes will appear to be identical, as well as why. The mathematics in this seemingly simple activity is elegant and perfect for younger pupils.

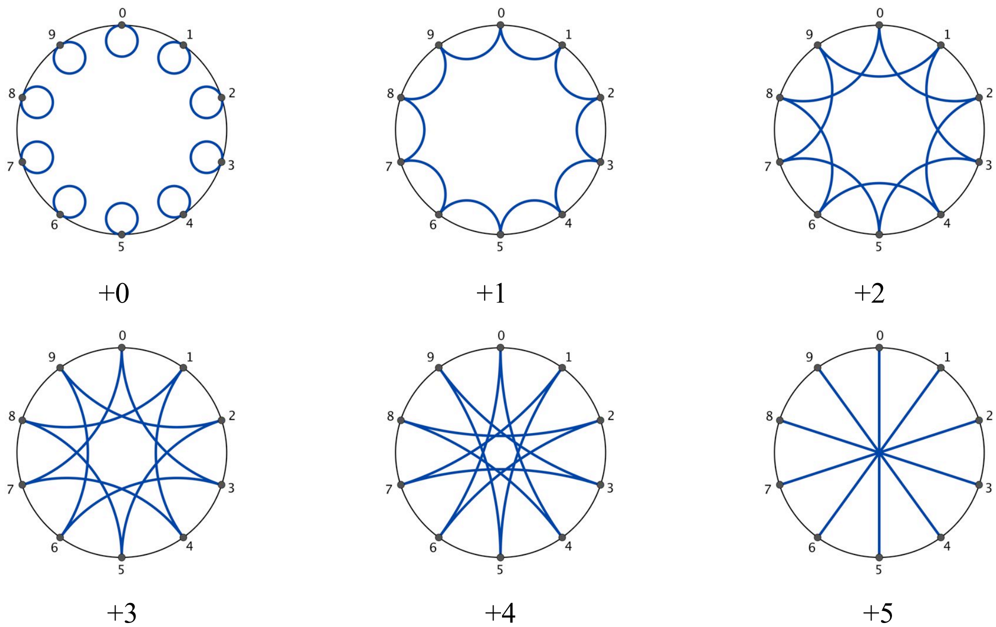


Figure 7: Addition runes “plus 0” to “plus 5”

Skip-counting Runes

Another variation for younger children is the construction of skip-counting runes. The “skip three” rune for example, shows the series 0, 3, 6, 9, 12, 15, 18, ... etc. (with the same wrap-around idea as before). The “skip three” rune looks just like the “plus three” rune above, but the “skip 2” rune is different from the “plus 2” rune (Figure 7). Which other skip runes are different from their corresponding plus runes, and why?

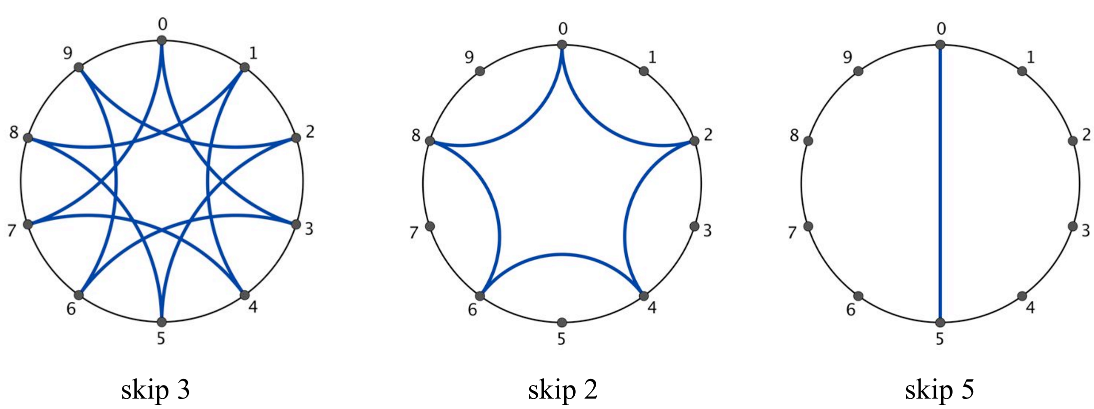


Figure 8: Skip-counting runes

Linear Function Runes

A whole constellation of runes is possible by combining multiplication and addition: choose a constant from 0-9 to multiply with the value of each point, and then add another constant from 0-9 to that result. Runes can be characterized by the linear equation $a\boldsymbol{x} + b$, where $\boldsymbol{x}$ is the input value on the circle and a and b are the two constants. The 100 possibilities are shown in Figure 10. The patterns within the rows and columns are intriguing and surprising. You are invited to puzzle over the figures – there are many secrets and surprises within this collection of patterns!

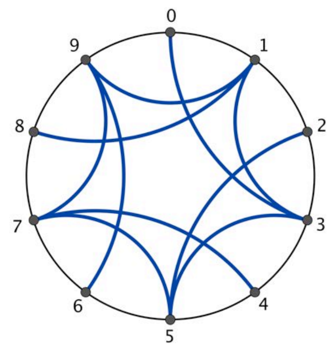


Figure 9: $6\boldsymbol{x} + 3$ rune

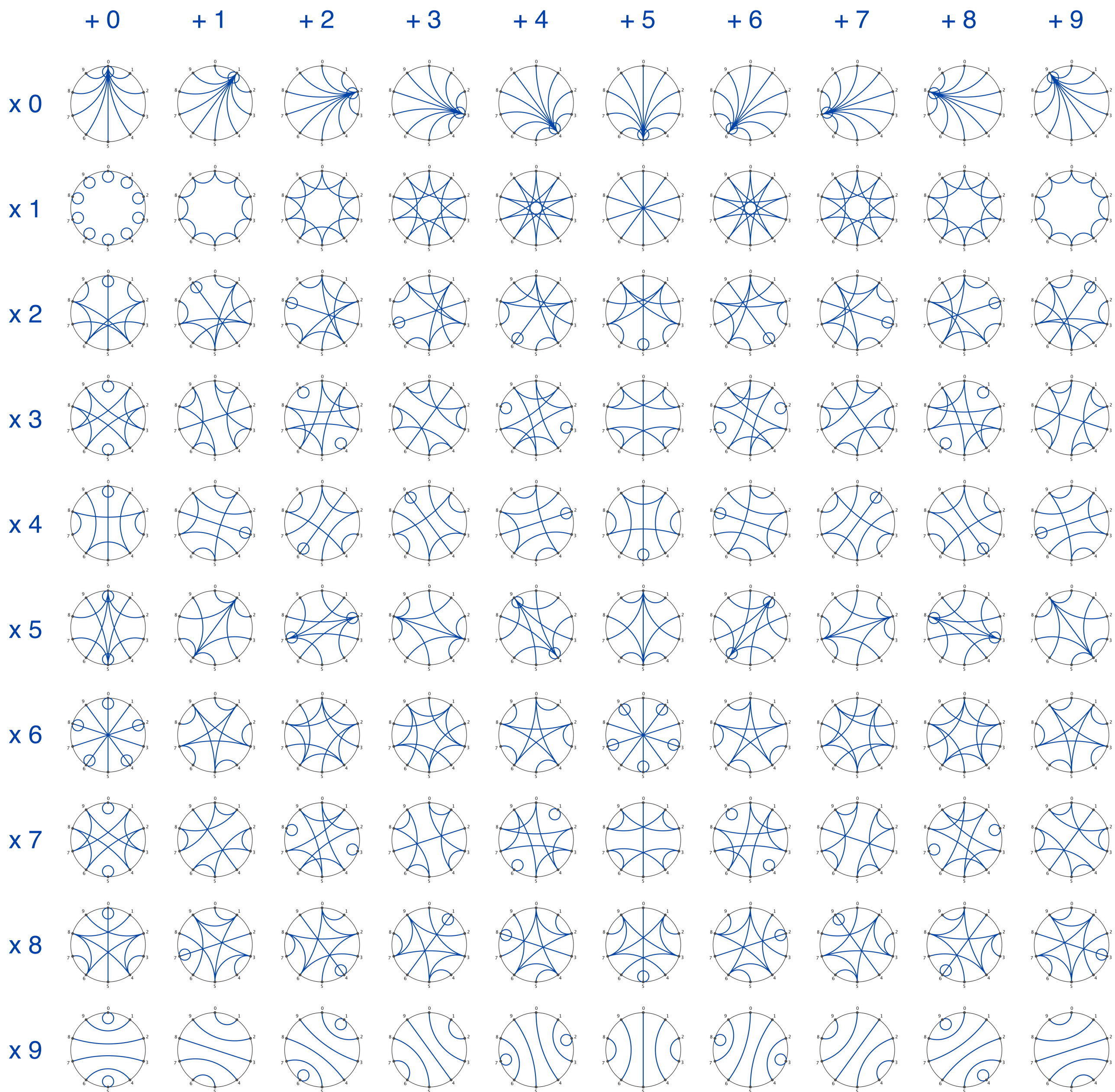


Figure 10: Complete table of runes for the linear function $a\boldsymbol{x} + b$

Other functions

Any function that produces integer outputs can produce math runes. Several simple polynomial functions are shown in figure 11. The first set, the family of $x^2 + b$ runes, are all different. This is not the case in the second set, the family of $x^3 + b$ runes; 8 of the runes are "partners" and share a pattern that is the same as another in the family apart from a rotation. What symmetries are to be found in the families of $x^4 + b$, $x^5 + b$, and so on?

Triangular numbers, tetrahedral numbers, the Fibonacci sequence, exponential numbers, prime numbers... there are many sequences and patterns to explore!

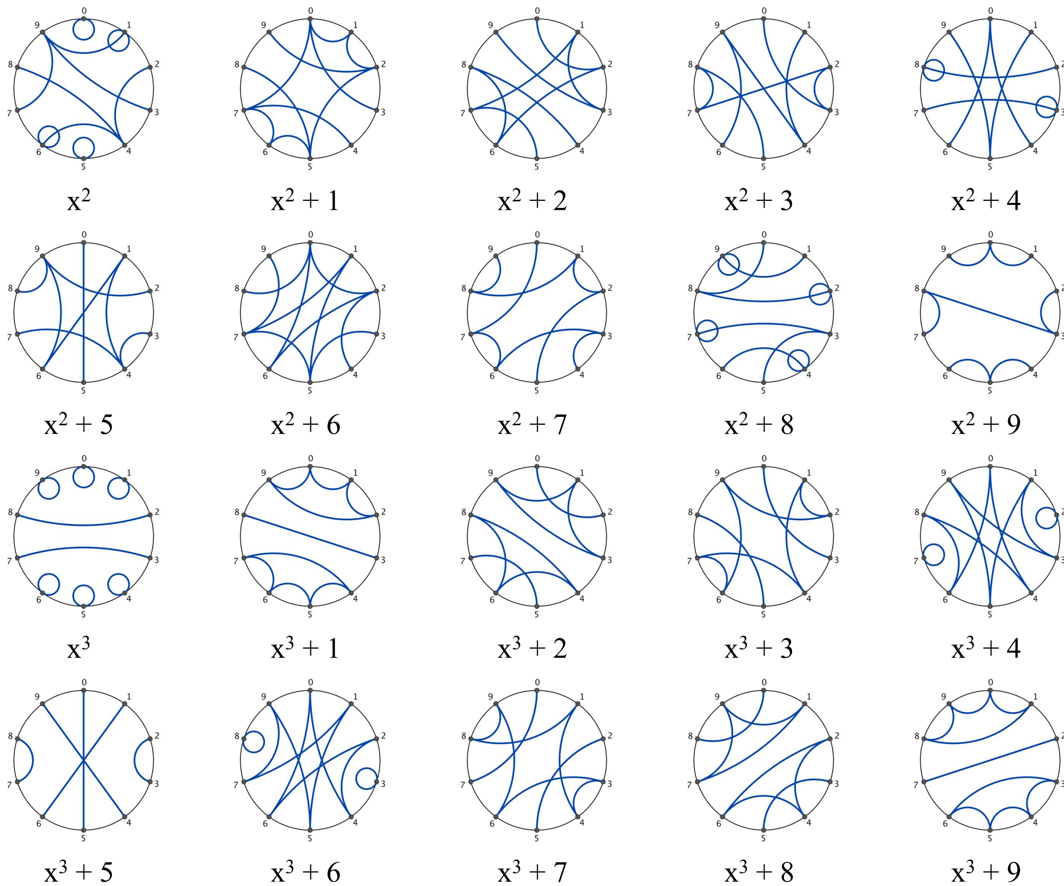
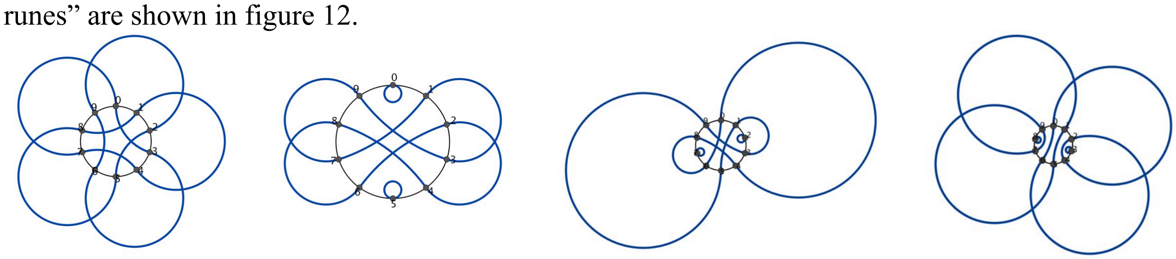


Figure 11: Selected polynomial functions

Flowers

Other pleasing forms can be made by constructing arcs both inside and outside of the circle. A few of these "flower runes" are shown in figure 12.

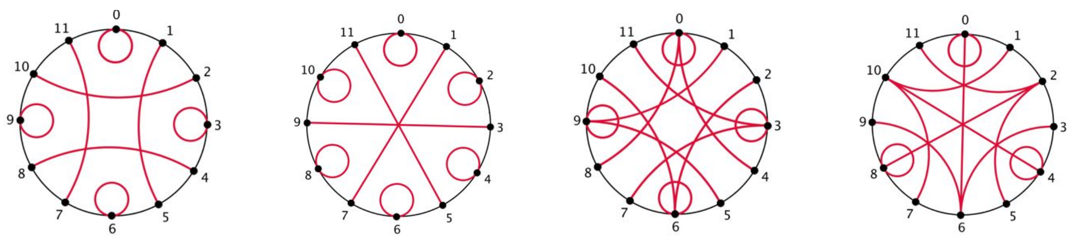


+ 3 | × 3 | $x^3 + 4$ | $x^3 + 6$

Figure 12: Flower runes

Different moduli

There are more worlds to explore with different moduli! Construction of these runes can help with practicing and understanding modular arithmetic. (One must be careful as finding the correct point from the output of a function is no longer as easy as looking at the last digit of the result!) There is also much to be learned by grouping families of runes according to their symmetries. In the selection of mod 12 multiplication runes shown in Figure 13, we see examples of 3-fold, 4-fold and 6-fold symmetry. How are the modulus, multiplier and order of symmetry related? You may wish to start by exploring simple examples with modular bases such as 3, 4 and 5.



× 5 × 7 × 9 × 10

Figure 13: A selection of mod 12 mulitplication runes

Resources

Some of the figures in this paper were made by hand and some were generated using Geogebra, a free geometry/ CAS application available at http://www.geogebra.org. In the resource section at http://mike-naylor.com you will also find a pdf template with blank math rune forms and the Geogebra file to create math runes automatically using your own formulas.

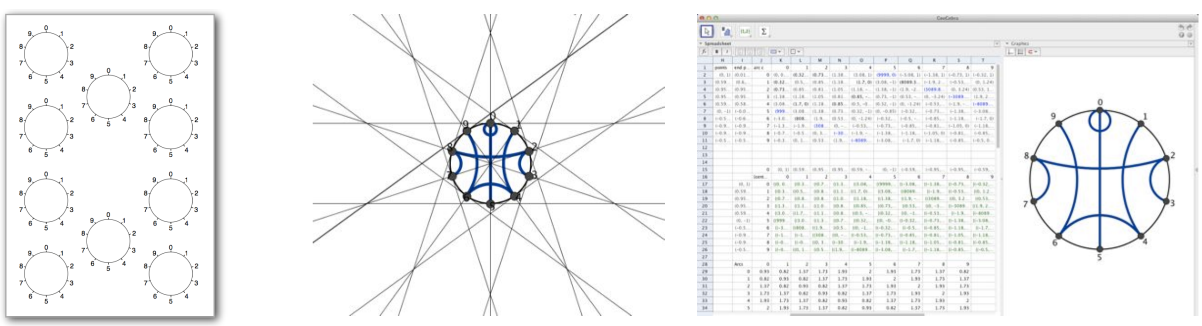
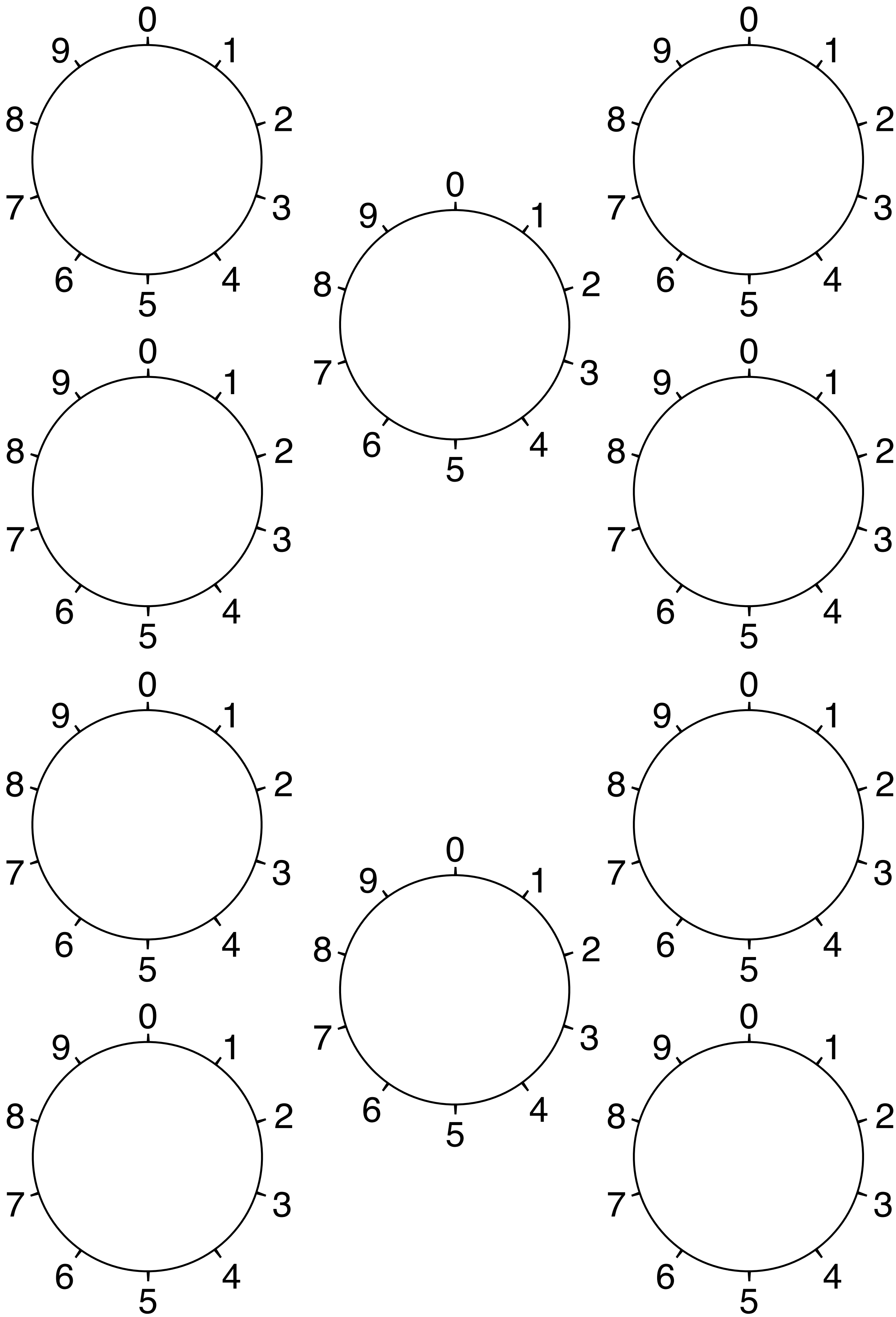


Figure 14: Math rune templates and Geogebra files are available at mike-naylor.com

Closing the Loop

Creating math runes is a rich and rewarding activity. The mathematical ideas are accessible for all, at many different levels, and touch upon a range of mathematical topics: basic operations, skip counting, modular arithmetic, sorting, symmetry, conjecture and testing of conjectures, functions, generalizations, and use of technology. In addition, the artwork created is lovely, open for further creative exploration, and highlights the beauty of mathematics.

We end our journey with the Elder Futhart rune *jera*. This rune is used to symbolize the completion of a cycle and thus seems to be especially appropriate in relation to our math runes. I hope you have enjoyed the cycles here, and wish you a pleasant voyage further!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